Physica 18D (1986) 405–409
North-Holland, Amsterdam

LYAPUNOV STABILITY CONDITIONS FOR RELATIVISTIC MULTIFLUID PLASMA

Darryl D. HOLM[a] and Boris A. KUPERSHMIDT[a,b]
[a] *Theoretical Division & Center for Nonlinear Studies, MS-B284, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*
[b] *The University of Tennessee Space Institute, Tullahoma, TN 37388, USA*

The Hamiltonian structure is given for ideal relativistic multifluid plasma dynamics in the laboratory frame with the Hamiltonian functional equaling the relativistic energy minus the mass energy. The noncanonical Poisson bracket for this system turns out to be the same as for the nonrelativistic multifluid plasma, but with dynamical variables replaced by their relativistic counterparts. New constants of the motion are then derived from the Hamiltonian structure and used as Lyapunov functionals for proving sufficient conditions for nonlinear stability of relativistic multifluid plasma equilibria. The nonrelativistic limit of the formulation is uniformly regular, and nonlinear Lyapunov stability conditions derived previously for a nonrelativistic multifluid plasma reemerge in that limit.

1. Introduction

As shown in [1, 2], the (special) relativistic Hamiltonian formalism of ideal fluid dynamics is a regular, structure-preserving deformation of the nonrelativistic theory with parameter c^{-2} where c is the speed of light. This regularity allows the extension of the Hamiltonian methods for Lyapunov stability analysis [3–5] to the relativistic case. Using this type of stability analysis, we deduce sufficient conditions for Lyapunov stability of relativistic multifluid plasmas (MFP) which limit (as c^{-2} tends to zero) to the corresponding conditions for the nonrelativistic case, treated in [4]. This result is illustrated for planar barotropic MFP, but it also holds in three dimensions and for ordinary compressible fluids without electromagnetic fields. For the planar case, the stability condition we obtain generalizes Rayleigh's classical inflection point criterion for stability of planar incompressible fluids, to the case of compressible relativistic plasmas.

An evolutionary system for variables $\psi(x,t)$ is a Hamiltonian system if it can be expressed as $\partial_t\psi = \{H, \psi\}$ for some Hamiltonian $H(\psi)$ and Poisson bracket $\{\cdot,\cdot\}$, which is bilinear, skew-symmetric, and satisfies the Jacobi identity. An equilibrium solution $\psi_e(x)$ is Lyapunov stable in the norm $\|\cdot\|$, if for every $\epsilon > 0$ there is a $\delta > 0$ such that for each solution $\psi = \psi_e + \delta\psi$ ($\delta\psi$ is a perturbation) satisfying $\|\delta\psi\| > \delta$ at some initial time, the norm of the perturbation satisfies $\|\delta\psi\| < \epsilon$ for all subsequent time (assuming the solution continues to exist).

The search for an appropriate norm $\|\cdot\|$ for Lyapunov stability is facilitated by having an evolutionary system in Hamiltonian form for which the Poisson bracket admits Casimir functions C, such that $\{C, F\} = 0$ for all $F(\psi)$. Note that H and $H + C$ generate the same evolution under the Poisson bracket, if C is a Casimir. If $H_C := H + C$ has a critical point for a certain equilibrium flow, i.e., $\delta H_C(\psi_e) = DH_C(\psi_e)\cdot\delta\psi = 0$, then the second variation

$$\delta^2 H_C(\psi_e) = D^2 H_C(\psi_e)\cdot(\delta\psi)^2$$

will be preserved by the linearized motion at ψ_e. This happens because $\frac{1}{2}\delta^2 H_C(\psi_e)$ is the Hamiltonian for the linearized evolution of infinitesimal perturbations, $\delta\psi$, as a Hamiltonian system in terms of the Poisson bracket linearized at ψ_e [5]. If $\delta^2 H_C(\psi_e)$ is definite in sign for a certain equilibrium, then conservation of the norm defined by

0167-2789/86/$03.50

$\|\delta\psi\|^2 = \delta^2 H_C(\psi_e)$ expresses Lyapunov stability of the equilibrium solution, under the linearized evolution of perturbations. If, in addition, $H_C(\psi_e + \delta\psi)$ is convex in the sense discussed in [3–5] for finite $\delta\psi$, then quadratic forms that bound the conserved quantity

$$\hat{H}_C = H_C(\psi_e + \delta\psi) - H_C(\psi_e) - DH_C(\psi_e)\cdot\delta\psi$$

also define a norm expressing Lyapunov stability of the equilibrium ψ_e for finite amplitude perturbations, i.e., then nearby displacements from equilibrium remain nearby, under the nonlinear evolution of the system, in the sense of the norm constructed from the bounds on $\hat{H}_C$. Usually, the Casimirs C contain arbitrary functions, which determine classes of equilibria as critical points of H_C. The properties of stable equilibria (i.e., the stability conditions) are then defined in terms of conditions on these functions.

The Lyapunov method generally gives sufficient conditions for stability, so if the quantity $\delta^2 H_C(\psi_e)$ is indefinite for some equilibrium flow, there is logically no implication about instability. However, in some cases [6] the loss of Lyapunov stability as the equilibrium properties change signals the onset of either instability, or some other interesting process, such as Arnold diffusion. Alternatively, the lack of this type of stability criterion in some cases results because the original problem is ill-posed [7].

2. Multifluid plasma (MFP)

As shown in [1, 2], relativistic fluid dynamics is a regular, structure-preserving deformation of the nonrelativistic theory. In particular, for MFP the nonrelativistic stability criteria, being a set of inequalities, should be regularly deformed under relativisation. We illustrate this for barotropic MFP flows in a domain $D \subset \mathbb{R}^2$ in the x, y plane. Analogous methods and conclusions apply in three dimensions for adiabatic MFP.

The physical variables in the laboratory frame are (suppressing species indices): ρ, mass density; $\boldsymbol{N}$, fluid momentum density; $\boldsymbol{E}$, electric field; and $\boldsymbol{B}$, magnetic field. For the motion to remain planar, each of the dependent variables $\{\rho, \boldsymbol{N}, \boldsymbol{E}, \boldsymbol{B}\}$ must be functions only of (x, y, t); $\boldsymbol{N}$ and $\boldsymbol{E}$ must lie in the x, y plane; and $\boldsymbol{B} = B\hat{z}$ must be directed normally to the plane, along $\hat{z}$. The velocity $\boldsymbol{v}$ is related to momentum density by $\boldsymbol{v} = \boldsymbol{N}/\gamma w \rho$ where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the relativistic factor and $w = 1 + (e_0 + P_0/\rho_0)/c^2$, with subscript 0 for quantities defined in the rest frame of the fluid, such as internal energy e_0, pressure P_0, and rest mass density ρ_0.

The planar MFP equations are

$$\partial_t B = -\hat{z}\cdot\operatorname{curl}\boldsymbol{E} = E_{1,2} - E_{2,1},$$

$$\partial_t \boldsymbol{E} = \nabla B \times \hat{z} - \sum a\rho\boldsymbol{v},$$

$$\partial_t \rho = -\operatorname{div}\rho\boldsymbol{v},$$

$$\partial_t(\gamma w \boldsymbol{v}) = \boldsymbol{v} \times B^*\hat{z} + a\boldsymbol{E} - \nabla(wc^2\gamma),$$

where $B^* := \hat{z}\cdot\operatorname{curl}(\gamma w \boldsymbol{v}) + aB$, the parameter a is the species charge-to-mass ratio, and Σ indicates summation over species, with species indices supressed. The static Maxwell equations,

$$\operatorname{div}\boldsymbol{B} = 0, \quad \operatorname{div}\boldsymbol{E} - \sum a\rho = 0,$$

are preserved by the dynamics, if they are assumed to be initially satisfied.

The Hamiltonian structure for the relativistic MFP equations is contained in [2] and [8]. For functionals I, J of $\{\rho, \boldsymbol{N}, \boldsymbol{E}, B\}$, the Poisson bracket $\{I, J\}$ is given by

$$\begin{aligned}\{I, J\} = -\sum \int \mathrm{d}x\,\mathrm{d}y \bigg\{ &\frac{\delta J}{\delta\rho}\operatorname{div}\left(\rho\frac{\delta I}{\delta \boldsymbol{N}}\right) \\ &+ \frac{\delta J}{\delta N_i}\left(N_j\partial_i + \partial_j N_i\right)\frac{\delta I}{\delta N_j} \\ &- a\rho B\hat{z}\cdot\frac{\delta J}{\delta \boldsymbol{N}} \times \frac{\delta I}{\delta \boldsymbol{N}} + a\rho\frac{\delta J}{\delta \boldsymbol{E}}\cdot\frac{\delta I}{\delta \boldsymbol{N}} \\ &+ \frac{\delta J}{\delta \boldsymbol{N}}\cdot\left(\rho\nabla\frac{\delta I}{\delta\rho} - a\rho\frac{\delta I}{\delta \boldsymbol{E}}\right)\bigg\} \\ &+ \int \mathrm{d}x\,\mathrm{d}y\left[\frac{\delta J}{\delta \boldsymbol{E}}\cdot\operatorname{curl}\left(\frac{\delta I}{\delta B}\hat{z}\right)\right. \\ &\left.- \frac{\delta J}{\delta B}\hat{z}\cdot\operatorname{curl}\frac{\delta I}{\delta \boldsymbol{E}}\right]. \end{aligned} \tag{1}$$

Using the relativistic energy as the Hamiltonian,

$$H=\sum\int \mathrm{d}x\,\mathrm{d}y\left[c^2\left(\sqrt{|N|^2/c^2+(\rho w)^2}-\rho\right)-P_0\right]+\int \mathrm{d}x\,\mathrm{d}y\left[\tfrac{1}{2}|E|^2+\tfrac{1}{2}B^2\right], \tag{2}$$

and the identities $c^2\rho_0\partial w/\partial\rho_0=\partial P_0/\partial\rho_0$, $\delta H/\delta N=v$, $\delta H/\delta\rho=c^2(w\gamma^{-1}-1)$, $\delta H/\delta E=E$, $\delta H/\delta B=B$, the MFP equations can be immediately recovered from $\partial_t\psi=\{H,\psi\}$ for $\psi\in\{\rho,N,E,B\}$ using (1). In [2] this Hamiltonian structure is shown to limit smoothly to the nonrelativistic case as c^{-2} tends to zero. This regular behavior enables us to calculate the relativistic extension of the stability analysis in [4].

We observe that the conserved quantities

$$C_F=\int \mathrm{d}x\,\mathrm{d}y\,\rho F(B^*/\rho), \tag{3}$$

where $B^*=\hat{z}\cdot\operatorname{curl}(\gamma w v)+aB$, are Casimirs for the Poisson bracket (1).

The stationary flows ρ_e, v_e, E_e, B_e, satisfy

$$v_e\times B_e^*\hat{z}=-aE_e+\nabla(c^2w_e\gamma_e),\quad E_e=-\nabla\phi, \tag{4}$$

$$\nabla B_e\times\hat{z}=\sum a\rho_e v_e,\quad \operatorname{div}\rho_e v_e=0, \tag{5}$$

from which it follows that

$$v_e\cdot\nabla(c^2w_e\gamma_e+a\phi)=0,\quad v_e\cdot\nabla(B_e^*/\rho_e)=0, \tag{6}$$

where $\phi(x)$ is the electrostatic potential. We take the relations (6) to mean that there exists a functional dependence $c^2w_e\gamma_e+a\phi=K(B_e^*/\rho_e)$ for a certain function $K(B_e^*/\rho_e)$ for each species. Thus, $v_e\times B_e^*\hat{z}=\nabla K(B_e^*/\rho_e)$ and consequently,

$$\rho_e v_e=\frac{K'(B_e^*/\rho_e)}{B_e^*/\rho_e}\hat{z}\times\nabla(B_e^*/\rho_e) \tag{7}$$

and

$$\frac{K'(B_e^*/\rho_e)}{B_e^*/\rho_e}=\frac{\rho_e v_e\cdot\hat{z}\times\nabla(B_e^*/\rho_e)}{|\nabla B_e^*/\rho_e|^2}. \tag{8}$$

Substitution of (7) into (5) gives

$$\nabla B_e=-\sum\frac{a}{B_e^*/\rho_e}\nabla K(B_e^*/\rho_e). \tag{9}$$

With these relations, one quickly establishes that $\delta H_C=0$ for stationary flows when $H_C=H+\Sigma C_F$. By this definition, we set

$$\begin{aligned}H_C=\sum\int_D \mathrm{d}x\,\mathrm{d}y\Big[&c^2\left[\sqrt{|N|^2/c^2+(\rho w)^2}-\rho\right]\\&-P_0+\rho F(B^*/\rho)+\lambda B^*\Big]\\&+\int_D \mathrm{d}x\,\mathrm{d}y\Big[\tfrac{1}{2}|E|^2+\tfrac{1}{2}B^2\\&+\phi(x)\left(-\operatorname{div}E+\sum a\rho\right)\Big],\end{aligned} \tag{10}$$

where λ is a constant. Using $\gamma w v=N/\rho$ and integrating by parts, we find

$$\begin{aligned}\delta H_C=\int_D \mathrm{d}x\,\mathrm{d}y\Big\{&\sum(v-\rho^{-1}\hat{z}\times\nabla F')\cdot\delta N\\&+\left[B+\sum a(F'+\lambda)\right]\delta B\\&+\sum\Big[c^2\left(\frac{w}{\gamma}-1\right)+a\phi+F-(B^*/\rho)F'\\&+\rho^{-2}N\cdot\hat{z}\times\nabla F'\Big]\delta\rho+(E+\nabla\phi)\cdot\delta E\Big\}\\&+\sum\oint_{\partial D}(\lambda+F')\delta(N/\rho)\cdot\mathrm{d}l,\end{aligned} \tag{11}$$

where $\delta(N/\rho):=\rho^{-1}\delta N-N\rho^{-2}\delta\rho$, the line element on the boundary is $\mathrm{d}l$, and we have neglected an electrostatic boundary term of no consequence. The first variation (11) vanishes by the stationary relations (4)–(7) provided the function $K(\xi)$ satisfies

$$K+F-\xi F'(\xi)=0, \tag{12}$$

or, equivalently,

$$F(\xi)=\xi\left(\int^{\xi}\frac{K(q)}{q^2}\,\mathrm{d}q+\text{const}\right), \tag{13}$$

where the constant c^2 has been absorbed into K. Then, each coefficient vanishes in the interior of D and the boundary term vanishes by choosing $\lambda + F'(B_e^*/\rho_e)|_{\partial D} = 0$, since B_e^*/ρ_e is constant on the boundary; by (6) and the boundary conditions $\boldsymbol{v}\cdot\hat{\boldsymbol{n}} = 0$, where $\hat{\boldsymbol{n}}$ is the unit normal vector on the boundary. Thus, H_C has a critical point for stationary flows when the Casimir function F is determined by (13).

Calculation of the second variation of H_C gives, using $\theta = (N^2/c^2 + (w\rho)^2)^{1/2} = w\rho\gamma$,

$$\begin{aligned}\delta^2 H_C &= D^2 H_C(\boldsymbol{N}_e, \rho_e, \boldsymbol{E}_e, \boldsymbol{B}_e)\cdot(\delta\boldsymbol{N}, \delta\rho, \delta\boldsymbol{E}, \delta\boldsymbol{B})^2 \\ &= \sum\int \mathrm{d}x\,\mathrm{d}y\Bigg\{\theta_e^{-1}|\delta\boldsymbol{N}|^2 \\ &\quad + \rho_e F''(B_e^*/\rho_e)[\delta(B^*/\rho)]^2 \\ &\quad + (\delta\rho)^2\left[\frac{\partial(c^2 w/\gamma)}{\partial\rho} - \frac{2|\boldsymbol{N}|^2}{\rho^2\theta}\right]_e \\ &\quad + 2\,\delta\rho\,\boldsymbol{N}_e\cdot\delta\boldsymbol{N}\Bigg[(\rho\theta)^{-1} \\ &\quad + \frac{\partial(c^2 w/\gamma)}{\partial|\boldsymbol{N}|^2} - \frac{1}{2}\theta^{-2}\frac{\partial\theta}{\partial\rho}\Bigg]_e \\ &\quad + (\boldsymbol{N}_e\cdot\delta\boldsymbol{N})^2\left[-2\theta^{-2}\frac{\partial\theta}{\partial|\boldsymbol{N}|^2}\right]_e\Bigg\} \\ &\quad + \int \mathrm{d}x\,\mathrm{d}y\left[|\delta\boldsymbol{E}|^2 + (\delta B)^2\right]. \end{aligned} \tag{14}$$

In the nonrelativistic limit, (14) reduces to the quadratic form in [4] which was shown to be positive definite, provided the equilibrium flow is everywhere subsonic and $F''(B_e^*/\rho_e) > 0$, or equivalently [by (8) and (13)], $\boldsymbol{v}_e\cdot\hat{\boldsymbol{z}}\times\nabla(B_e^*/\rho_e) > 0$.

In the relativistic case, the second variation (14) is positive definite and, hence, the corresponding equilibrium flow is linearly Lyapunov stable, when the following sufficient conditions are satisfied for each species:

$$\rho_e > 0, \tag{15}$$

$$F''(B_e^*/\rho_e) = \frac{\rho_e \boldsymbol{v}_e\cdot\hat{\boldsymbol{z}}\times\nabla(B_e^*/\rho_e)}{|\nabla(B_e^*/\rho_e)|^2} > 0, \tag{16}$$

$$\left[\frac{\partial(c^2 w/\gamma)}{\partial\rho} - \frac{2|\boldsymbol{N}|^2}{\rho^2\theta}\right]_e \geq 0, \tag{17}$$

$$\frac{\partial|\boldsymbol{v}_e|^2}{\partial|\boldsymbol{N}_e|^2} \geq 0, \tag{18}$$

$$\begin{aligned}&\left(|\boldsymbol{N}|^2\theta^{-1}\frac{\partial|\boldsymbol{v}|^2}{\partial|\boldsymbol{N}|^2}\left[\frac{\partial(c^2 w/\gamma)}{\partial\rho} - \frac{2|\boldsymbol{N}|^2}{\rho^2\theta}\right]\right)_e \\ &\quad \geq \left[\frac{1}{\rho\theta} + \frac{\partial(c^2 w/\gamma)}{\partial|\boldsymbol{N}|^2} - \frac{1}{2}\theta^{-2}\frac{\partial\theta}{\partial\rho}\right]_e^2 |\boldsymbol{N}_e|^2. \end{aligned} \tag{19}$$

Condition (15) is the physical condition of positive mass density, which implies $\theta_e > 0$, as well. Condition (16) is the geometrical condition that $\boldsymbol{v}_e, \hat{\boldsymbol{z}}, \nabla(B_e^*/\rho_e)$ form a right-handed triad. Condition (17) is the relativistic subsonic condition for each species. Condition (18) requires that the magnitude of each species velocity be an increasing function of the corresponding momentum density. Lastly, the condition (19) represents an additional, essentially-relativistic requirement for stability, which is not present in the nonrelativistic case (in the nonrelativistic limit, (19) reduces to the trivial inequality $0 \geq 0$). Details of the calculations outlined here and explicit applications of the stability conditions (15)–(19) are given in [9]. In the case of plane-parallel relativistic flow, the stability condition (16) generalizes Rayleigh's classical inflection point criterion for stability of planar flow of an incompressible fluid to the problem of relativistic charged flow in a planar diode with applied magnetic field.

The relativistic functional H_C in (10) can be made convex in the sense of [3–5] for sufficiently small c^{-2}; this is again because relativisation of fluid dynamics is a regular deformation. Therefore, solutions satisfying the relativistic stability criteria (17)–(19) and $F''(\xi) \geq \alpha > 0$ for all ξ and constant α, are nonlinearly Lyapunov stable, i.e., are stable for finite amplitude perturbations, for sufficiently small c^{-2}.

Acknowledgements

Sponsored by the United States Department of Energy (DOE) under contract W-7405-ENG-36 and DOE Office of Basic Energy Sciences, Department of Applied Mathematics, and by the National Science Foundation.

References

[1] D.D. Holm and B.A. Kupershmidt, Relativistic fluid dynamics as a Hamiltonian system, Phys. Lett. 101A (1984) 23–26.

[2] D.D. Holm and B.A. Kupershmidt, Relativistic chromohydrodynamics and Yang–Mills–Vlasov plasma, Phys. Lett. 105A (1984) 225–228.

[3] V.I. Arnold, Conditions for nonlinear stability of stationary curvilinear flows in an ideal fluid, Sov. Math. Dokl. 162 (1965) 773–7; On an a priori estimate in the theory of hydrodynamical stability, Am. Math. Soc. Transl. 79 (1969) 267–9.

[4] D.D. Holm, Stability of planar multifluid plasma equilibria by Arnold's method, Contemp. Math. 28 (1984) 25–50.

[5] D.D. Holm, J.E. Marsden, T. Ratiu and A. Weinstein, Nonlinear stability conditions and a priori estimates for barotropic hydrodynamics, Phys. Lett. 98A (1983) 15–21; Nonlinear stability of fluid and plasma equilibria, Phys. Rep. 123 (1985) 1–116.

[6] H.D.I. Abarbanel, D.D. Holm, J.E. Marsden and T. Ratiu, Richardson number criterion of the nonlinear stability of three dimensional stratified flows, Phys. Rev. Lett. 52 (1984) 2552–2555; Nonlinear stability analysis of stratified, incompressible, ideal fluid flow, Phil. Trans. Roy. Soc. (London) (to appear).

[7] D.D. Holm and B.A. Kupershmidt, Hydrodynamics and electrohydrodynamics of adiabatic multiphase fluids and plasmas, Int. J. Multiphase Flow (to appear).

[8] Z.R. Iwinski and L.A. Turski, Canonical theories of systems interacting electromagnetically, Lett. Appl. Sci. and Eng. 4 (1976) 179–191.

[9] D.D. Holm and B.A. Kupershmidt, Lyapunov stability of relativistic fluids and plasmas, Phys. Fluids (to appear).